

Universal spectral correlations from the ballistic σ model

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We consider the semiclassical ballistic σ model as an effective theory describing the quantum mechanics of classically chaotic systems. Specifically, we elaborate on close analogies to the recently developed semiclassical theory of quantum interference in chaotic systems and show how semiclassical “diagrams” involving near action degenerate sets of periodic orbits emerge in the field theoretical description. We further discuss how the universality phenomenon (i.e., the fact that individual chaotic systems behave according to the prescriptions of random matrix theory) can be understood from the perspective of the field theory.

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I. INTRODUCTION

In recent years, significant progress has been made in understanding the semiclassical basis of universality in quantum chaos. In a sequence of steps [1–6], the theory of action correlations and periodic orbits has advanced to a stage, where the full information stored in the spectral correlation functions of random matrix theory (RMT) has become accessible by semiclassical methods. An important element in this development has been the establishment of a cross-link between semiclassics and the so-called zero-dimensional σ model (which in turn had long been known [7] to be in one-to-one correspondence to RMT): under conditions of global hyperbolicity and for low energies (energies lower than the inverse \hbar/t_E of the so-called Ehrenfest time, t_E), individual contributions to the semiclassical expansion of spectral correlation functions become fully universal in that they depend on combinatorics and topology of the underlying orbit correlations, but not on system specific details. Each such term has a corresponding contribution to the loop expansion of the zero-dimensional σ model around one of its saddle points. From a certain perspective, the semiclassical analysis of spectral correlations has, thus, been tantamount to a reduction of the full microscopic information stored in the Green functions of an individual chaotic system down to the core data encapsulated in the zero-dimensional σ model.

In this paper, we discuss an alternative reduction scheme, which is similar in spirit, but methodologically different. Our starting point will be the observation that the zero-dimensional σ model affords an alternative interpretation, viz. as the globally uniform mean field limit of the ballistic σ model. The ballistic σ model in its variant considered here is an effective field theory of chaotic systems, obtained from a microscopic parent theory [8–10] by projection onto the sector of fields fluctuating on length scales $\gtrsim \hbar$. We will present various arguments to the effect that this projection captures the essential physics of the problem. The resulting effective field theory is defined on shells of conserved energy in classical phase space. Its dynamics is controlled by the—equally classical—Liouville operator. Quantum mechanics enters the theory through a feature known as “noncommutativity.” In practice, noncommutativity implies that points in phase space can be defined only up to a maximal resolution set by the Planck cell. (Notice that the typical linear extensions of

Planck cells are of $\mathcal{O}(\hbar^{1/2})$, much bigger than the cutoff length of the theory.) Below we will explore the conspiracy of classical hyperbolicity and Planck cell resolution in the long time dynamics of chaotic systems.

Specifically, we will show that a perturbative expansion of the model leads to structures that are remarkably similar, if not equivalent, to those arising in periodic orbit theory: taking the first quantum correction to the spectral correlation function (the “Sieber-Richter term” [2]) as an example, we will recover information on the phase space available to individual families of periodic orbits, and the corresponding action correlations. The actual integrals describing these corrections turn out to be identical to those of periodic orbit theory. This finding suggests a quantitative identification of the Feynman diagrams of the theory with the action-correlated orbit pairs of semiclassical analysis. (However, an extension of the calculation to higher orders in perturbation theory, necessary to substantiate this claim, is beyond the scope of the present paper.)

Going beyond perturbation theory, we will demonstrate that for correlation energies smaller than the inverse of the Ehrenfest time nonuniform fluctuations in phase space are effectively suppressed. The resulting theory of uniform fluctuations (viz. the zero-dimensional σ model) predicts universal behavior, in agreement with the predictions of RMT. Field theory thus provides a comparatively simple way of understanding the basis of the Bohigas-Gianonni-Schmid [11] universality hypothesis, alternative to the explicit summation over infinite hierarchies of periodic orbits [6]. The computational efficiency of the “mean field plus fluctuations” approach to universality arguably represents the most important advantage of the present formalism.

The rest of the paper is organized as follows: in Sec. II we review the semiclassical ballistic σ model. In Secs. III and IV, respectively, we explore the field theory approach to universality, and the connections to periodic orbit theory. A number of issues relating to the derivation of the theory will be addressed in Appendix A.

II. BALLISTIC σ MODEL

We wish to explore correlations in the level density, $\rho(E)$, of globally hyperbolic (chaotic) quantum systems, as characterized by the two point correlation function $R_2(s)$

$\equiv \Delta^2 \langle (\rho(E + \frac{\Delta s}{2\pi}) - \langle \rho(E + \frac{\Delta s}{2\pi}) \rangle) (\rho(E) - \langle \rho(E) \rangle) \rangle$. Here, $\langle \cdots \rangle$ denotes averaging over E over an interval $[E_0 - E_{\text{av}}/2, E_0 + E_{\text{av}}/2]$, where $E_{\text{av}} \ll E_0$. The goal is to show that for energies $\omega = s\Delta/\pi \lesssim \hbar t_E^{-1}$ the function $R_2(s)$ approaches the spectral correlation functions of RMT. Here, $t_E \equiv \lambda^{-1} \ln(c^2/\hbar)$ is the Ehrenfest time, λ the dominant Lyapunov exponent of the system, and c^2 some classical reference scale of dimension “action” whose specific value will not be of much relevance [12].

We represent the spectral correlation function in terms of a replica generating functional. (Choosing the replica variant of the theory is motivated by its high suitability to perturbative calculations [4]; it is a matter of a straightforward redefinition of the field target space to upgrade the formalism to a supersymmetric field theory.)

$$R_2(s) = -\frac{1}{2} \text{Re} \lim_{R \rightarrow 0} \frac{1}{R^2} \partial_s^2 Z(s), \quad (1)$$

where R is the number of replicas and the generating function is defined as

$$Z(s) \equiv \int DT \exp(-S[T]),$$

$$S[T] = \frac{\beta\pi\hbar}{2\Delta} \int_{\Gamma} (dx) \text{tr} \left(T^{-1} * \Lambda \{H, T\} + \frac{is\Delta}{2\pi\hbar} \Lambda T^{-1} * \Lambda T \right). \quad (2)$$

Equation (2) has the status of an effective field theory in classical phase space, obtained from a microscopic parent theory—the energy averaged field integral representations of the microscopic Green functions [8–10]—by elimination of field configurations fluctuating on short scales $< \hbar$. In Appendix A we will argue that Eq. (2) relates to the microscopic formalism in much the same way the semiclassical Gutzwiller sum relates to the microscopic Feynman path integral.

The integration variables in (2), $T(\mathbf{x}) = \{T^{\alpha\alpha'}(\mathbf{x})\}$ are matrix valued fields defined on shells $\Gamma = \{\mathbf{x} | H(\mathbf{x}) = E_0\}$ of constant energy in classical phase space. Here, $\mathbf{x} \equiv (\mathbf{q}, \mathbf{p})$ where \mathbf{q} and \mathbf{p} are coordinates and momenta, respectively, $H(\mathbf{x})$ is the Hamiltonian function of the system, and the integral over the energy shell is normalized to unity, $\int_{\Gamma} (dx) = 1$. For time reversal and spin rotation invariant systems (orthogonal symmetry class, $\beta=1$), the “internal” structure of the matrices $T^{\alpha\alpha'}$ is described by a composite index $\alpha = (a, r, t)$, where $a = +/ -$ discriminates between the advanced and retarded sector of the theory, $r=1, \dots, R$ is a replica index, and $t=1, 2$ accounts for the operation of time reversal. Time reversal symmetry reflects in the relation $\sigma_2^{\text{tr}} T^T \sigma_2^{\text{tr}} = T^{-1}$, where σ_i are Pauli matrices and the superscript “tr” indicates action in time reversal space. For time reversal noninvariant systems (unitary symmetry, $\beta=2$) no time-reversal structure exists and $\alpha = (a, r)$. In either case, the matrices T carry a coset space structure in the sense that configurations T and TK are identified if $[K, \Lambda] = 0$, where $\Lambda = \sigma_3^{\text{ar}}$ and “ar” stands for action in advanced/retarded space.

The fluctuation behavior of the fields T in (2) is governed by the classical Liouville operator $\{H, \}$ (where $\{, \}$ is the Poisson bracket). Quantum mechanics enters the problem through the back door, viz. by the presence of Moyal products “*” in (2). The Moyal product between two phase space functions A and B is defined as

$$(A * B)(\mathbf{x}) = \int \frac{d^f \mathbf{x}_1}{(\pi\hbar)^f} \frac{d^f \mathbf{x}_2}{(\pi\hbar)^f} e^{\frac{2i}{\hbar} \mathbf{x}_1^T I \mathbf{x}_2} A(\mathbf{x} + \mathbf{x}_1) B(\mathbf{x} + \mathbf{x}_2), \quad (3)$$

where $I \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic unit matrix. (For all practical purposes, the definition (3) will be more convenient than the standard representation [13], $(A * B)(\mathbf{x}) = \exp(\frac{i\hbar}{2} \partial_{\mathbf{x}'}^T I \partial_{\mathbf{x}}) A(\mathbf{x}') B(\mathbf{x})|_{\mathbf{x}=\mathbf{x}'}$.) The presence of the Moyal product implies that (a) all quantities appearing in the theory get effectively averaged (smoothed) over Planck cells. Relatedly, (b) the generator of classical time evolution $\{H, \}$ acts on *distributions* smooth in phase space on scales $\sim \hbar^f$, rather than on mathematical points. In the following sections we will explore how the interplay of hyperbolic dynamics ($\{H, \}$) and Planck cell smearing (*) determines the output of the theory.

III. UNIVERSAL LIMIT

We wish to explore the behavior of the theory (2) for correlation energies $\omega \ll \hbar t_E^{-1}$ much smaller than the inverse of the Ehrenfest time. For simplicity, we will consider systems with broken time reversal invariance throughout this section.

For energies $\omega \gg \Delta \Leftrightarrow s \gg 1$, the partition function $Z(s)$ may be evaluated by perturbative methods. To prepare the perturbative expansion of the action, we introduce the rational parameterization

$$T = \mathbb{1} + W, \quad W = \begin{pmatrix} & -B^\dagger \\ B & \end{pmatrix}, \quad (4)$$

where the block structure is in advanced/retarded space and the generators B are $R \times R$ complex matrices. Substitution of this representation then leads to a series $S[B, B^\dagger] = \sum_{n=1}^{\infty} S^{(2n)}[B, B^\dagger]$, where $S^{(n)}$ is of n th order in B and B^\dagger . Specifically,

$$S^{(2)}[B, B^\dagger] = t_H \int_{\Gamma} (dx) \text{tr} (B^\dagger \mathcal{L}_\omega B), \quad (5)$$

where

$$\mathcal{L}_\omega = -i\omega/\hbar - \{H, \} \quad (6)$$

and the Moyal product stars have been omitted for notational simplicity. Higher order terms in the action contain traces over (Moyal) products of matrices $BB^\dagger \dots B^\dagger$. Due to the Planck-cell “averaging” inherent to the product (3), the Wick contraction of individual matrix elements of B and B^\dagger will generate expressions

$$P_\omega(\check{\mathbf{x}}, \check{\mathbf{x}}') = \int_0^\infty dt e^{i\omega t} P_t(\check{\mathbf{x}}, \check{\mathbf{x}}'), \quad (7)$$

where

$$P_\omega \equiv (-i\omega/\hbar - \{H, \cdot\})^{-1}, \quad P_t \equiv e^{t\{H, \cdot\}} \quad (8)$$

are the (retarded) Liouville propagators in the energy and time representation, respectively, \mathbf{x} and \mathbf{x}' are arbitrary points in phase space and

$$f(\check{\mathbf{x}}) \equiv \frac{1}{\hbar^f} \int_{\hbar^f} d^f \mathbf{y} f(\mathbf{x} + \mathbf{y})$$

is symbolic notation for coordinate averaging over a Planck cell. We interpret $P_t(\cdot, \check{\mathbf{x}}')$ as the dynamical evolution of a smooth phase space distribution of extension \hbar^f and centered around \mathbf{x}' . It has been rigorously shown by methods of symbolic dynamics [14] that for times $t < t_E$, these distributions are centered around the classical trajectory through \mathbf{x}' , i.e., the dynamics is approximately described by the Liouville evolution of \mathbf{x}' . However, beyond $t \approx t_E$, the distribution rapidly (over time scales comparable to t_{mix}) crosses over to a uniform distribution in phase space. These structures are summarized in the ansatz

$$P_t(\check{\mathbf{x}}, \check{\mathbf{x}}') \equiv \begin{cases} \Omega \delta(\check{\mathbf{x}} - \check{\mathbf{x}}'(t)), & t \lesssim t_E, \\ 1, & t > t_E, \end{cases} \quad (9)$$

where $\Omega = \frac{\hbar^f}{\Delta}$ is the volume of the energy shell, and the normalization is such that for our unit-normalized phase space integral, $\int_\Gamma (d\mathbf{x}) P_t(\check{\mathbf{x}}, \check{\mathbf{x}}') = 1$.

For energy scales $\omega < \hbar t_E^{-1}$ the dominant contribution to the time integral in (7) comes from large times $t_E < t \lesssim \hbar \omega^{-1}$. One expects that in this regime phase space fluctuations will have effectively relaxed to spatial uniformity. To describe the damping of inhomogeneous modes in more explicit terms we follow a prescription formulated by Kravtsov and Mirlin [15], then in the context of diffusive systems: employing the ansatz

$$T(\mathbf{x}) = T_0(1 + W_P(\mathbf{x})) \quad (10)$$

we isolate the inhomogeneous contents of the fields T . Here, $T_0 = \text{const}$ describes the zero mode sector and

$$W_P(\mathbf{x}) \equiv W(\mathbf{x}) - W_Q(\mathbf{x}), \quad W_Q(\mathbf{x}) \equiv \int_\Gamma (d\mathbf{x}) W(\mathbf{x})$$

is a projection onto purely fluctuating field configurations: $\int (d\mathbf{x}) W_P(\mathbf{x}) = 0$. Substitution of Eq. (10) into the action and expansion to second order in $W_P = \begin{pmatrix} -B_P^\dagger \\ B_P \end{pmatrix}$ generates the decomposition

$$S[T_0, W_P] = S_0[Q_0] + S^{(2)}[B_P, B_P^\dagger] + S_c[T_0, B_P, B_P^\dagger], \quad (11)$$

where

$$S_0[Q_0] = \frac{i\omega\pi}{2\Delta} \text{tr}(Q_0\Lambda), \quad Q_0 = T_0\Lambda T_0^{-1} \quad (12)$$

is the zero mode action, and

$$S^{(2)}[B_P, B_P^\dagger] = t_H \int_\Gamma (d\mathbf{x}) \text{tr}(B_P^\dagger \mathcal{L}_0 B_P), \quad (13)$$

the quadratic action of the fluctuating fields governed by the generator $\mathcal{L}_0 = \mathcal{L}_{\omega=0} = \delta - \{H, \cdot\}$, $\delta \searrow 0$ of the time integrated dynamics. Finally,

$$S_c[T_0, B_P, B_P^\dagger] = -\frac{i\omega t_H}{2\hbar} \int_\Gamma (d\mathbf{x}) \text{tr}((T_0^{-1} \Lambda T_0)^{11} B_P^\dagger B_P - (T_0^{-1} \Lambda T_0)^{22} B_P B_P^\dagger)(\mathbf{x}), \quad (14)$$

where the superscripts refer to advanced-retarded space. Following Ref. [15], we expand in S_c , retaining only contributions of minimal order in ω :

$$S_0[Q_0] \rightarrow S_{\text{eff}}[Q_0] = S_0[Q_0] + \langle S_c[T_0, B_P, B_P^\dagger] \rangle - \frac{1}{2} \langle (S_c[T_0, B_P, B_P^\dagger])^2 \rangle,$$

where $\langle \dots \rangle \equiv \int D(B_P, B_P^\dagger) \exp(-S^{(2)}[B_P, B_P^\dagger]) (\dots)$. While the contraction of the B 's in the first contribution to the second line vanishes in the replica limit, the second term generates the effective action

$$S_{\text{eff}}[Q_0] = S_0[Q_0] - \frac{\omega^2}{16\hbar^2} (\text{tr}(Q_0\Lambda))^2 \times \int_\Gamma (d\mathbf{x})(d\mathbf{x}') (P_P)_0(\mathbf{x}, \mathbf{x}') (P_P)_0(\mathbf{x}', \mathbf{x}), \quad (15)$$

where, again, the subscript P stands for projection onto the fluctuating sector:

$$(P_P)_0(\mathbf{x}, \mathbf{x}') \equiv \left(1 - \int_\Gamma (d\mathbf{x})\right) \left(1 - \int_\Gamma (d\mathbf{x}')\right) P_0(\mathbf{x}, \mathbf{x}') = \int_0^\infty dt (\delta(\mathbf{x} - \mathbf{x}'(t)) - \Omega^{-1}), \quad (16)$$

and in the last line we have switched to a time representation. Notice that the Liouville propagator in Eq. (15) is evaluated on single phase space points, \mathbf{x} and \mathbf{x}' , rather than on Planck cells $\check{\mathbf{x}}$ and $\check{\mathbf{x}}'$. This is because the phase space integral of the Moyal product of two operators collapses to the ordinary product [16], i.e.,

$$\int_\Gamma (d\mathbf{x}) \text{tr}((T_0^{-1} \Lambda T_0)^{11} B_P^\dagger * B_P)(\mathbf{x}) = \int_\Gamma (d\mathbf{x}) \text{tr}((T_0^{-1} \Lambda T_0)^{11} B_P^\dagger(\mathbf{x}) B_P(\mathbf{x})).$$

Substituting the result Eq. (16) into Eq. (15), we obtain

$$S_{\text{eff}}[Q_0] = S_0[Q_0] - \frac{t_E \Omega}{16\hbar^2} (\omega \text{tr}(Q_0\Lambda))^2 \times \int_0^\infty dt \int_\Gamma (d\mathbf{x})(\delta(\mathbf{x} - \mathbf{x}(t)) - \Omega^{-1}). \quad (17)$$

According to Eq. (17), the correction term is given by the

integrated weight of periodic orbits $\mathbf{x} \rightarrow \mathbf{x}$ of duration t , minus the total phase volume. Using that [17]

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T dt \int_{\Gamma} (d\mathbf{x}) \Omega \delta(\mathbf{x} - \mathbf{x}(t)) = 1,$$

one concludes that this term cancels, i.e., to lowest order in the expansion, fluctuations of the inhomogeneous modes do not change the universal zero-mode action. At higher orders in the expansion, we are met with expressions [cf. Eq. (7)]

$$\begin{aligned} (P_P)_\omega(\check{\mathbf{x}}, \check{\mathbf{x}}') &= \int_0^\infty dt e^{i\frac{\omega t}{\hbar}} \left(1 - \int_{\Gamma} (d\mathbf{x}) \right) \left(1 - \int_{\Gamma} (d\mathbf{x}') \right) P_t(\check{\mathbf{x}}, \check{\mathbf{x}}') \\ &\simeq \int_0^{t_E} dt e^{i\frac{\omega t}{\hbar}} (\Omega \delta(\mathbf{x} - \mathbf{x}'(t)) - 1), \end{aligned}$$

where Eq. (9) was used. The important point here is the truncation of the time integral at t_E . This means that at n th order in the expansion in fluctuation propagators, P_P , corrections $\sim (\omega t_E/\hbar)^n$ will be generated. Our analysis above shows that terms with $n < 3$ cancel, i.e., corrections to the zero mode action can arise only at $\mathcal{O}(\omega t_E \hbar^{-1}) \sim \mathcal{O}(t_E/t_H)^3$,

$$S_{\text{eff}}[Q_0] = S_0[Q_0] + \mathcal{O}(\omega t_E/\hbar)^{n \geq 3}.$$

This observation is consistent with the periodic orbit analysis of Ref. [18] where it has been shown that the leading correction to the universal result scales as $(\omega t_E/\hbar)^3$.

Summarizing, our analysis shows that at correlation energies $\omega < \hbar/t_E$ inhomogeneous fluctuations get effectively damped out and the field theory collapses to an integral over the zero mode. This means that RMT results (plus weak corrections in the parameter $(\omega t_E/\hbar)^{n \geq 3}$) will be obtained for spectral correlation functions, and other observables probing the long time behavior of the system. Our ‘‘mode damping’’ approach to universality is complementary to semiclassical analysis [6], where the spectral correlation functions are constructed explicitly, by summation over infinite hierarchies of periodic orbits. The connection between these approaches will be explored in the next section.

IV. PERTURBATIVE EQUIVALENCE TO SEMICLASSICS

Having shown that the ballistic σ model crosses over to the universal zero dimensional model at low energies, we now approach the problem from a different perspective. We will perform a straightforward perturbative expansion around the high energy saddle point Λ to elaborate on parallels between the field theory (2) and the periodic orbit approach to spectral correlations. Focusing on the lowest order (in τ , see below) quantum corrections to the spectral form factor (the so-called Sieber-Richter term [2]), we will show that the two formalisms are structurally similar, to an extent that the correlations of individual periodic orbits can be reproduced from the field theory formalism.

Throughout this section, we will consider time reversal invariant systems, i.e., $\beta=1$.

A. Semiclassics

For the benefit of nonexpert readers, we begin with a brief review of recent semiclassical results. Consider the Fourier transform of the spectral correlation function,

$$K(\tau) \equiv \frac{1}{\pi} \int ds R_2(s) e^{-2is\tau}. \quad (18)$$

The RMT result for the $\beta=1$ form factor reads as

$$K(\tau) = \begin{cases} 2\tau - \tau \ln(1 + 2\tau), & \tau < 1 \\ 2 - \tau \ln\left(\frac{2\tau + 1}{2\tau - 1}\right), & \tau \geq 1. \end{cases} \quad (19)$$

Specifically, the short time expansion of the spectral form factor in $\tau \ll 1$ (corresponding to the expansion of the spectral correlation function in $\omega \gg \Delta$) starts as

$$K(\tau) = 2\tau - 2\tau^2 + \dots \quad (20)$$

In semiclassics one aims to reconstruct that expansion from the Gutzwiller double sum over periodic orbit pairs (α, α') ,

$$K(\tau) = \left\langle \sum_{\alpha\alpha'} A_\alpha A_{\alpha'}^* e^{i(S_\alpha - S_{\alpha'})} \delta\left(\tau - \frac{t_\alpha + t_{\alpha'}}{2t_H}\right) \right\rangle, \quad (21)$$

where $\langle \dots \rangle$ is an average over orbit energies (and a small interval of orbit times) and A_α , S_α , and t_α are the stability amplitude, the action, and the time of traversal of orbit α , respectively.

The first term in (20) obtains from Berry’s diagonal approximation,

$$K^{(1)}(\tau) \equiv 2\tau,$$

i.e., an approximation that retains only identical orbits $\alpha = \alpha'$ and mutually time reversed $\alpha = \bar{\alpha}'$ orbits, and uses the Hannay-Ozorio de Almeida [19] sum rule $\langle \sum_\alpha |A_\alpha|^2 \delta(\tau - t_\alpha/t_H) \rangle = \tau$ to determine the weight of the remaining orbit sum.

The structure of the leading order (in τ) quantum corrections to the spectral form factor was first identified by Aleiner and Larkin [1]: two orbits differing in the presence or absence of a small angle self-intersection in configuration space interfere to provide a stable contribution to the double sum (cf. Fig. 1.) For the sake of later comparison to the σ model formalism, we briefly review the quantitative computation of this contribution [2] in the invariant language of hyperbolic phase space coordinates [20] for a system with $f=2$ degrees of freedom.

Imagine a Poincaré section cutting through the encounter region where the participating orbit segments have their (avoided) self-intersection (cf. Fig. 2). Choosing one of the piercing points, \mathbf{x} , as the origin of a local coordinate system (cf. inset of Fig. 2), we introduce $2f-1=3$ energy shell coordinates (t, u, s) , where t is a timelike coordinate running along the trajectory through \mathbf{x} , and u/s are coordinates along the locally unstable/stable directions (the existence of the latter guaranteed by the assumption of global hyperbolicity.) One may then show [2,20] that the contribution of all orbit pairs containing a single encounter region is given by

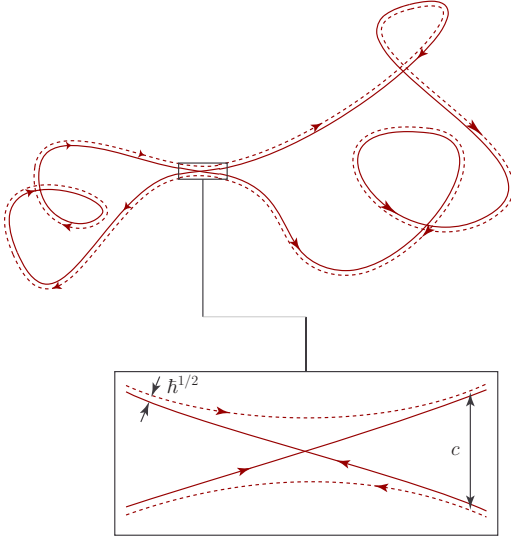


FIG. 1. (Color online) Cartoon of an orbit pair contributing to the first quantum correction to the spectral form factor.

$$K^{(2)}(\tau) = 2 \frac{\tau^2}{h} \int dudse^{i\hbar^{-1}us} \frac{t - 2t_{\text{enc}}(u,s)}{2t_{\text{enc}}(u,s)} = -2\tau^2. \quad (22)$$

Here, $t_{\text{enc}}(u,s) = \lambda^{-1} \ln(c^2/us)$ is the time of traversal through the encounter region, λ the (self-averaging) Lyapunov exponent of the system, and c^2 some classical action scale. We note that (i) the dominant contributions to the integral are of order $us \sim \hbar$ implying that $t_{\text{enc}}(u,s) \sim \lambda^{-1} \ln(c^2/\hbar)$ is of the order of the Ehrenfest time, (ii) the product us is an invariant of the motion (Liouville's theorem), which means that the choice of the Poincaré section entering the construction is arbitrary, and (iii) the only surviving contribution to the integrand in (22) are of order $\mathcal{O}(t_{\text{enc}}(u,s)^0)$. (Terms containing t_{enc} , either from the numerator, or the expansion of the denominator can be shown to effectively oscillate to zero in the semiclassical limit [3,4,6].)

In later work, the perturbative analysis of action correlations has been extended first to cubic and then to infinite order [3–5] in the τ expansion. Employing an unconventional representation of the two-point correlation function [6] it has

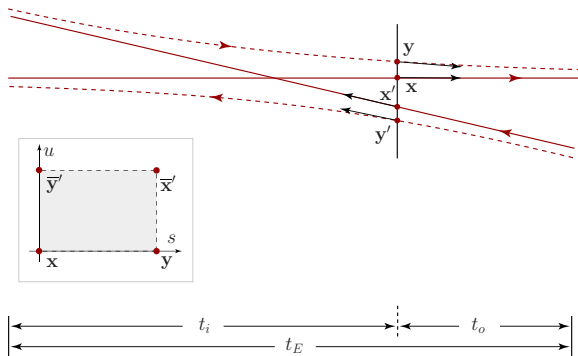


FIG. 2. (Color online) Phase space representation of the encounter region. Here the bar in \bar{x} , \bar{y} stands for time reversal, as discussed in the text.

also been possible to reconstruct the bottom part of Eq. (19), which is not accessible in terms of a straightforward series expansion of the conventional two-Green function representation.

B. Field theory

Equation (22) encapsulates the phase volume available to individual pairs of interfering trajectories (preexponent) and the corresponding action correlation (exponent). Extending the analysis of Ref. [16], we will show below that the same information is stored in the effective action (2). Although our discussion will be restricted to first order in perturbation theory in τ (the extension of the analysis to higher orders in perturbation theory is left for future work), it suggests a general equivalence between periodic orbit theory and the perturbative expansion of the σ model.

In the time reversal invariant case considered here, the blocks B in (4) are $(2R \times 2R)$ matrices subject to the time reversal symmetry condition $B^\dagger = -\sigma_2^T B^T \sigma_2^T$.

Substitution of this representation into the action (2) generates the first order correction to the spectral correlation function

$$R_2(s)^{(2)} = -\text{Re} \lim_{R \rightarrow 0} \frac{t_H}{(2R)^2} \partial_s^2 \int_{\Gamma} (d\mathbf{x}) \langle \text{tr}(B^\dagger B B^\dagger \mathcal{L}_\omega B) \rangle,$$

where we have omitted the Moyal product stars for the sake of notational simplicity, and averaging is over the quadratic action, $\langle \dots \rangle \equiv \int \mathcal{D}(B^\dagger, B) e^{-S^{(2)}[B, B^\dagger]} (\dots)$. Doing the Gaussian integrals over matrix elements of B , we obtain (cf. Ref. [16] for technical details),

$$R^{(2)}(s) = \text{Re} \frac{\Omega^2}{t_H} \partial_s^2 \int_{\Gamma} (d\mathbf{x}) \int \frac{d\mathbf{y}_1 d\mathbf{y}_2}{(2\pi\hbar)^{2(f-1)}} e^{\frac{i}{\hbar} \mathbf{y}_2^T \mathbf{y}_1} P_\omega(\mathbf{x} + \mathbf{y}_1/2, \mathbf{x} - \mathbf{y}_1/2) \mathcal{L}_{\omega, \mathbf{x} + \mathbf{y}_2/2} P_\omega(\mathbf{x} + \mathbf{y}_2/2, \mathbf{x} - \mathbf{y}_2/2).$$

Here, $\bar{\mathbf{x}} \equiv (\mathbf{q}, -\mathbf{p})^T$ is the time reversed of the phase space point $\mathbf{x} = (\mathbf{q}, \mathbf{p})^T$, the integrals over \mathbf{y}_1 and \mathbf{y}_2 represent the generalization of the Moyal product (3) to the trace of a product of four operators, and the subscript \mathbf{x} in $\mathcal{L}_{\omega, \mathbf{x}}$ indicates on which coordinate of the propagator $P_\omega(\mathbf{x}, \mathbf{x}')$ the derivatives of the Liouville operator act. To make further progress, we Fourier transform the expression above whereupon it transmutes to the first quantum correction to the spectral form factor,

$$K^{(2)}(\tau) = -2\tau^2 \Omega^2 \int_{\Gamma} (d\mathbf{x}) \int \frac{d\mathbf{y}_1 d\mathbf{y}_2 e^{\frac{i}{\hbar} \mathbf{y}_2^T \mathbf{y}_1}}{(2\pi\hbar)^{2(f-1)}} \int_0^t dt' P_{t-t'}(\mathbf{x} + \mathbf{y}_1/2, \mathbf{x} - \mathbf{y}_1/2) \mathcal{L}_{t', \mathbf{x} + \mathbf{y}_2/2} P_{t'}(\mathbf{x} + \mathbf{y}_2/2, \mathbf{x} - \mathbf{y}_2/2). \quad (23)$$

Here, $\mathcal{L}_t \equiv \partial_t - \{H, \}$ and P_t is the propagator in time representation [cf. Eq. (8)].

Equation (23) contains the essence of the process shown in Fig. 1: $K^{(2)}(\tau)$ is expressed in terms of a product of two ergodic return probabilities at times t' and $t-t'$, respectively. The fine structure of the encounter dynamics (the boxed re-

gion in the figure) is encapsulated in the integral over phase space coordinates $\mathbf{y}_{1,2}$. As we will see momentarily, this integral limits the integral over t' to a window $[t_{\text{enc}}, t-t_{\text{enc}}]$, again in analogy to semiclassical analysis.

To proceed we introduce a coordinate system that has \mathbf{x} as its origin and $\mathbf{x}+\mathbf{x}' \leftrightarrow (r, u, s)$, where r is the coordinate of \mathbf{x}' along the classical trajectory running through \mathbf{x} and u and s parameterize the components of \mathbf{x}' in the locally stable and unstable direction around \mathbf{x} (see Fig. 2). We assume [21] that the function $P_t(\overline{\mathbf{x}+\mathbf{y}/2}, \mathbf{x}-\mathbf{y}/2)$ exhibits the following characteristics: (i) $P_t(\overline{\mathbf{x}+\mathbf{y}/2}, \mathbf{x}-\mathbf{y}/2) = P_t(u)$ depends only on the unstable component of \mathbf{y} . Indeed, this component controls the rate at which the trajectory starting at $\mathbf{x}-\mathbf{y}/2$ deviates from that through \mathbf{x} . This deviating component (rather than the approaching component, s , determines the spatial and temporal structure of the Lyapunov region. Conversely, $P_t(\overline{\mathbf{x}+\mathbf{y}/2}, \mathbf{x}-\mathbf{y}/2)$ depends only on s . (ii) In order for $P_t(\overline{\mathbf{x}+\mathbf{y}/2}, \mathbf{x}-\mathbf{y}/2)$ to become nonvanishing, the two trajectories through $\mathbf{x}-\mathbf{y}/2$ and $\overline{\mathbf{x}+\mathbf{y}/2}$ must have left the Lyapunov region around \mathbf{x} (cf. the “large scale picture” Fig. 1). This takes a time of order $2 \times t(u)$, where

$$t(u) \equiv \lambda^{-1} \ln(c/|u|)$$

accounts for “half” of the encounter time. Additional to this time, some time of classical duration, roughly of order t_{mix} , is required to “tie” the outgoing and incoming trajectory segments to a closed link. These assumptions are summarized in the ansatz

$$P_t(\overline{\mathbf{x}+\mathbf{y}/2}, \mathbf{x}-\mathbf{y}/2) = \Omega^{-1} \tilde{\Theta}(t-2t(u)), \quad (24)$$

where $\tilde{\Theta}$ is a smeared step function interpolating between zero and unity over a time interval of order t_{mix} . Neither the detailed structure of $\tilde{\Theta}$, nor the exact value of $t(u)$ are of further relevance. We note, however, that the postulated independence of P_t of the “longitudinal” coordinate, r , implies stationarity of the long time probability distribution $P_{t>t(u)}$ under the Hamiltonian flow, $\{H, P_{t>t(u)}\} = 0$.

Using that $\mathcal{L}_t P_t \approx \partial_t P_t$, the first quantum correction becomes ($f=2$)

$$\begin{aligned} K^{(2)}(\tau) &= -\frac{2\tau^2\Omega^2}{(2\pi\hbar)^2} \int_0^t dt' \int du_1 ds_1 du_2 ds_2 e^{\frac{i}{\hbar}(u_1 s_2 - s_1 u_2)} \\ &\quad \times P_{t-t'}(s_1) \partial_{t'} P_{t'}(u_2) \\ &= \frac{\tau^2\Omega^2}{2\pi\hbar} \int_0^t dt' \int dudse^{\frac{i}{\hbar}us} P_{t-t'}(s) \partial_{t(u)} P_{t'}(u) \\ &\approx \frac{\tau^2}{2\pi\hbar} \int dudse^{\frac{i}{\hbar}us} \partial_{t(u)} (t-2t(u)-2t(s)) \\ &= \frac{2\tau^2}{2\pi\hbar} \int dudse^{\frac{i}{\hbar}us} \partial_{t_{\text{enc}}(u,s)} \frac{t-2t_{\text{enc}}(u,s)}{2} \\ &= \frac{2\tau^2}{2\pi\hbar} \int dudse^{\frac{i}{\hbar}us} \frac{t-2t_{\text{enc}}(u,s)}{2t_{\text{enc}}(u,s)}, \end{aligned} \quad (25)$$

in agreement with the semiclassically derived expression (22). In the first line we used that, under the conditions stated

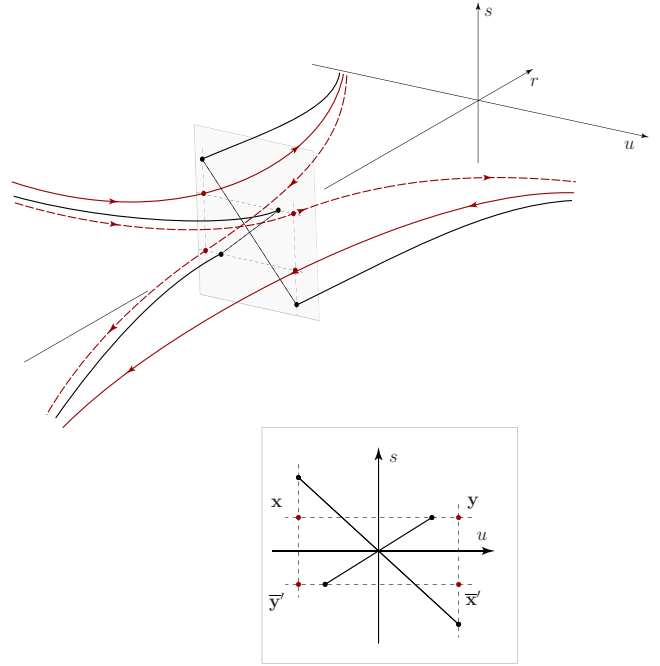


FIG. 3. (Color online) Assignment of propagator coordinates (lines terminating in thick points) to Feynman amplitudes (lines continued through the Poincaré section).

above, the long time probability distribution is invariant under the action of the Liouville operator. In the fourth line, we introduced the full encounter time, $t_{\text{enc}} = t(u) + t(s)$, and the last equality is based on the above mentioned vanishing of any power $(t_{\text{enc}}(u, s))^n$ upon integration against the oscillatory kernel $\sim \exp(i\hbar^{-1}us)$.

The derivation above demonstrates that field theory and semiclassics, respectively, allocate the same phase space volume to single encounter processes. We finally show that the field theoretical expression (25) indeed affords an interpretation in terms of individual periodic orbits. To this end, we consider a Poincaré section through the encounter, as shown schematically in Fig. 3. The four lines terminating in the two pairs of connected points in the phase space plane represent segments of classical trajectories beginning at $\mathbf{x}+\mathbf{y}_{1,2}/2$ and $\mathbf{x}-\mathbf{y}_{1,2}/2$, respectively, i.e., idealized classical trajectories corresponding to the arguments of the field theory propagators in (23). Each of these points has a stable (s) and an unstable (u) coordinate. The fact that the propagators P are retarded implies that we can distinguish between “incoming” and “outgoing” terminal points. Now, consider the behavior of a trajectory carrying the stable coordinate of an incoming terminal propagator point, and the unstable coordinate of a terminal point of the other propagator pair (cf. inset of Fig. 3). The trajectory running through this reference point will interpolate between the propagator stretches involved, and is naturally interpreted as part of a closed loop. The analogous definition of three more points (see inset of Fig. 3) plus associated trajectory stretches leads to the identification of a periodic orbit and its topologically distinct partner orbit. The action difference between these two orbits is but the product us . This construction implies an interpretation of the field theory expression (25) purely in terms of periodic orbits. The

correspondence between the two formalisms is established in terms of symplectic coordinates (u, s, t) characterizing *individual* periodic orbits: apparently, the action (2) encapsulates fine structure information about hyperbolic dynamics and action correlations in ergodic quantum systems.

V. SUMMARY AND DISCUSSION

In this paper we have considered the semiclassical ballistic σ model as an effective theory of quantum chaos. Defined in classical phase space, the semiclassical σ model is a field theory whose dynamics is driven by the classical Liouville operator. Quantum mechanics enters through a structure known as noncommutativity. In practice, noncommutativity (i) limits the maximal resolution of the theory to structures of the order of the Planck cell, and (ii) leads to the appearance of characteristic phases, which play a role analogous to the action correlations of periodic orbit theory.

We have shown in perturbation theory that the σ model describes spectral correlations in far reaching analogy to semiclassics. The fact that semiclassics and field theory attribute the same weight to *individual* orbit correlations in phase space [cf. Eqs. (22) and (25)] makes one suspect that the σ model fully encapsulates the information carried by the Gutzwiller double sum. However, this expectation has not been proven beyond first order in perturbation theory.

Perhaps most importantly, we have shown that the semiclassical σ model provides for an efficient description of the crossover to the universal regime of RMT correlations at energies below the inverse of the Ehrenfest time. Adapting a technique originally developed by Kravtsov and Mirlin to describe spectral correlations in disordered systems, we saw that in chaotic systems—in contrast to disordered systems—quadratic inhomogeneous fluctuations do not give relevant corrections to the universal action. Extending this approach to higher orders in the inhomogeneous modes, a systematic quantitative estimate for the corrections to RMT spectral statistics in individual chaotic systems can, in principle, be obtained. Methodologically, this way of approaching the universality phenomenon is complementary to (and arguably more economical than) the infinite order summations over correlated orbit pairs of Refs. [3,4].

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APPENDIX A: REGULARIZATION

The σ model discussed in the foregoing sections has the status of an *effective* theory, obtained from an underlying “bare” theory by elimination of rapid fluctuations. In this section we discuss the status of that reduction. Comparing to semiclassics, we will argue that the projection onto an effective field theory resembles the passage from the Feynman path integral to the semiclassical Gutzwiller sum.

The action of the bare ballistic σ model [8–10] is given by

$$S_{\text{qu}}[T] = -\frac{i\beta\pi}{2E_{\text{av}}} \text{tr} \left(T^{-1} \Lambda [\hat{H}, T] - \frac{\Delta S}{2\pi} \Lambda T^{-1} \Lambda T \right), \quad (\text{A1})$$

where \hat{H} is the Hamiltonian operator of the system and tr is a trace over Hilbert space, projected to an energy strip of thickness E_{av} centered around E_0 . The integration variables $T = \{T_{\mathbf{q}\mathbf{q}'}^{\alpha\alpha'}\}$ are operators in a product Hilbert space spanned by real space coordinates, \mathbf{q} , and “internal” coordinates, α . (The internal structure of the matrices $T_{\mathbf{q}\mathbf{q}'}$ has been discussed in Sec. II above.)

The action $S_{\text{qu}}[T]$ is obtained from the rigorous functional integral representation of the two level correlation function after (a) a saddle point approximation (which generates the nonlinear constraint $Q^2=1$, where $Q=T\Lambda T^{-1}$) and (b) first order expansion in commutators $[\hat{H}, T^{-1}]$. As we will argue in Appendix A below, these two approximations are largely immaterial.

1. Semiclassical model

Preliminary contact with the (semi)classical contents of the theory is made by switching to a Wigner representation. Upon Wigner transformation of Hilbert space operators $A_{\mathbf{q}_1, \mathbf{q}_2} \rightarrow A(\mathbf{x}) \equiv \int d\Delta \mathbf{q} e^{i\hbar^{-1}\Delta \mathbf{q} \cdot \mathbf{p}} A_{\mathbf{q}+\Delta \mathbf{q}/2, \mathbf{q}-\Delta \mathbf{q}/2}$, the operators $T_{\mathbf{q}, \mathbf{q}'} \rightarrow T(\mathbf{x})$ transmute to the fields in classical phase space discussed above [22]. Further, $\text{tr}(\cdots) \rightarrow \int_{\Gamma_{E_{\text{av}}}} d^f \mathbf{x}(\cdots)$ becomes an integral over a phase space energy shell $\Gamma_{E_{\text{av}}}$, centered around E_0 and of thickness E_{av} . We are thus led to the phase space representation [16]

$$S_{\text{ps}}[T] = -\frac{i\beta\pi}{2E_{\text{av}}} \int_{\Gamma_{E_{\text{av}}}} \frac{d^f \mathbf{x}}{h^f} \text{tr} \left(T^{-1} * \Lambda [H^*, T] - \frac{\omega}{2} \Lambda T^{-1} * \Lambda T \right), \quad (\text{A2})$$

where the asterisks stand for Moyal products, as usual.

Importantly, the field theory (A2) does not contain mechanisms inhibiting the buildup of rapid fluctuations. The action $S[T]$ of fields fluctuating on quantum scales $\sim \hbar^\alpha$, $\alpha \geq 1$ in classical phase space is qualitatively different from the semiclassical action considered above: for fields fluctuating transverse to the energy layer of thickness $E_{\text{av}} \sim \hbar/t_0 \sim \hbar$ reduction of the phase space integral to an integral over a single energy “shell” is no longer possible. Further, the series expansion of the Moyal product

$$[H^*, T] = i\hbar\{H, T\} + \hbar^3 \mathcal{O}(\partial^3 H \partial^3 T)$$

shows that for fields fluctuating on scales $\sim \hbar^\alpha$, $\alpha \geq 1$, the approximation of the quantum commutator by the Poisson bracket is no longer permissible.

A safe way to eliminate those rapid field fluctuations is by adding a weak random potential to the Hamiltonian. An ensemble average over that randomness will generate a second order differential operator which may be tailored so as to effectively remove fast field fluctuations. As discussed in

Ref. [23] a “quantum random potential” $V_q \sim \hbar$, vanishing in the classical limit (and parametrically in \hbar weaker than the “regulator” suggested by Aleiner and Larkin [1,24]) suffices to remove field fluctuations on phase space scales $\sim \hbar$.

Less rigorously, one may argue that the action S of rapidly fluctuating fields will lead to highly oscillatory exponents $\exp(iS)$ which likely tend to average out: a glance at Eq. (2) shows that the largest contributions to the action of the problem, set by the largest value of the correlation energy $\omega \sim \hbar/t_{\text{mix}}$, scale as $\Delta^{-1} \times \hbar$. However, fields fluctuating on scales $\sim \hbar^\alpha$, $\alpha \geq 1$ will lead to contributions of $\Delta^{-1} \times \hbar^{n(1-\alpha)}$, where the odd index n designates the order of the Moyal expansion. One may argue that in the semiclassical limit, these terms generate strong phase cancellations which render the contribution of strongly fluctuating fields effectively meaningless. (For a caveat in this argumentation, see Appendix A 2 below.)

We note that the above phase cancellation argument resembles the logics inherent to the stationary phase derivation of the Gutzwiller trace formula from the Feynman path integral; there, too, avoidance of rapid fluctuations is the dominant principle. More specifically, Gutzwiller’s trace formula is based on a stationary phase approximation to the Feynman path integral in S/\hbar (where S stands for the typical value of a classical action.) This approximation effectively averages over fine structures on scales \hbar^α , $\alpha \geq 1$. Relatedly, the semiclassical analysis of spectral correlations will be oblivious to the averaging of the system Hamiltonian over “quantum” perturbations $V_q \sim \hbar$. (Not affecting the classical dynamics, such perturbations merely change the phases weighing individual periodic orbits. In the evaluation of the Gutzwiller double sum of spectral correlations, these phases cancel out [25].)

Summarizing, the semiclassical σ model (2) considered in the main text obtains as a projection of the “quantum” σ model (A1) onto the sector of fields fluctuating on scales $> \hbar$. This projection may be effected either by averaging the system over a quantum random potential of strength $\sim \hbar$, or by alluding to the prospected irrelevance of strongly fluctuating actions in the semiclassical limit (i.e., *ad hoc* restriction of the functional integral to fields fluctuating on scales $> \hbar$).

One thus reduces the microscopic theory (equivalent to the full Feynman path integral representation of Green function correlators) to an effective theory (likely equivalent to the Gutzwiller approximation of the Feynman path integral).

2. Zero modes

While the above phase cancellation arguments apply to “generic” field configurations, there is one family of rapidly fluctuating configurations that deserves separate consider-

ation: the quantum action (A1) is nullified by a large number of *zero modes* [26] fluctuating at scales of the order of the Fermi wavelength. Choosing a representation in terms of eigenfunctions, $\hat{H}|\psi_a\rangle = \epsilon_a|\psi_a\rangle$, wherein $T = \{T_{a,a'}^{\alpha\alpha'}\}$ and $[\hat{H}, T] = (\epsilon_a - \epsilon_{a'})\{T_{a,a'}^{\alpha\alpha'}\}$, we conclude that there exist $N \sim E_{\text{av}}/\Delta$ zero modes $T_{\alpha\alpha'}^{aa'}$ whose action is controlled only by the energy contribution, $S_{\text{qu}}[T_{aa}] = \frac{i\beta\pi\omega}{4E_{\text{av}}} \text{tr}(\Lambda T_{aa}^{-1} \Lambda T_{aa})$. Upon Wigner transformation, the zero mode operators turn into N zero mode functions of the action (A2), rapidly fluctuating in classical phase space. These modes are certainly not “unphysical.” In the case of time reversal noninvariant systems, one may indeed verify [27] that the integration over T_{ab} leads to the formally exact eigenvalue decomposition

$$R_2(\omega) = \frac{\Delta^2}{E_{\text{av}}|E_0 - \epsilon_{a,b}| \leq E_{\text{av}}/2} \sum \delta(\omega - (\epsilon_a - \epsilon_b)). \quad (\text{A3})$$

What makes the rigorous derivation of Eq. (A3) possible is a mathematical principle known as “semiclassical exactness” [28]. (Incidentally, we note that the possibility to obtain an exact representation of the spectral correlation function from the quantum ballistic σ model shows that the approximations on which the derivation of the latter is based—saddle point approximation and first order expansion in the quantum commutator—become exact in the limit $E_{\text{av}}/\Delta \rightarrow \infty$.)

In spite of its formal correctness, Eq. (A3) is, of course, useless in practice (much like the evaluation of the Feynman path integral in an exact basis of eigenfunctions, formally an exact alternative to the semiclassical stationary phase approximation, would be pointless).

The conservative way to remove the zero modes is, again, by averaging over weak randomness. Indeed, semiclassical exactness represents a very delicate structure; even miniscule changes in the action will spoil the exact cancellation of all non-Gaussian fluctuations on which Eq. (A3) is based. (Although we are lacking a rigorous justification, we believe that averaging over a random potential whose inverse scattering time, or level broadening, is as weak as $\hbar \tau_q^{-1} \sim \Delta$ will suffice to effectively remove the zero modes.) It is, then, favorable to switch to a Wigner phase space representation, and classify fluctuations along the lines of the semiclassical analysis of Appendix /A 1. Alternatively one may avoid averaging over randomness and accept the presence of zero modes—after all, the integration over these modes produces a meaningful, if useless result. Mapping the integral onto a phase space representation restricted to slowly fluctuating modes one may deliberately sacrifice the exact information stored in the zero modes in return for a semiclassically meaningful approximation scheme.

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